Inflation and accelerated expansion TeVeS cosmological solutions

Luz Maria Diaz-Rivera, Lado Samushia, and Bharat Ratra
Department of Physics, Kansas State University, 116 Cardwell Hall, Manhattan, KS 66506, USA
(Dated: January 2006 KSUPT-06/1)

We find exact exponentially expanding and contracting de Sitter solutions of the spatially homogeneous TeVeS cosmological equations of motion in the vacuum TeVeS model and a power law accelerated expanding solution in the presence of an additional ideal fluid with equation of state parameter $-5/3 < \omega < -1$. A preliminary stability analysis shows that the expanding vacuum solution is stable, while in the ideal fluid case stability depends on model parameter values. These solutions might provide a basis for incorporating early-time inflation or late-time accelerated expansion in TeVeS cosmology.

PACS numbers: 98.80.-k, 98.80.Jk, 04.20.Jb

I. INTRODUCTION

For more than seven decades hypothetical dark matter has been postulated to explain observed large-scale velocities that are larger than expected on the basis of Newton's second law of motion, Newton's inverse-square law of gravitation, and the assumption that cosmological structures under investigation are in gravitational equilibrium, [1, 2, 3]. Great effort has been devoted to the search for direct evidence of dark matter, [4], and although there have been advances there is no conclusive direct evidence for dark matter.

Observations disfavor the alternate possibility that the gravitational force falls off slower with distance than Newton's inverse-square law predicts and that there is not a significant amount of dark matter (see Secs. IV.A.1 and IV.B.13 of Ref. [5]).

Another explanation — modified Newtonian dynamics (MOND) — has been proposed by Milgrom [6]. In MOND, Newton's second law, $\vec{a} = -\nabla\Phi_N$, is modified to

$$\mu(|\vec{a}|/a_0)\vec{a} = -\nabla\Phi_N. \tag{1}$$

Here $\vec{a}$ is acceleration, a_0 is an acceleration scale, Φ_N is the Newtonian potential, and $\mu(x)$ is a function which satisfies $\mu(x) \approx 1$ when $x \gg 1$, and $\mu(x) \approx x$ when $x \ll 1$. From large-scale galactic data it has been estimated that $a_0 \approx 1 \times 10^{-8}\ \mathrm{cm\,s^{-2}}$, so in the solar system where accelerations are large compared to a_0 the usual Newtonian law is recovered. Equation (1) and this choice of $\mu(x)$ was proposed to explain the observed flat rotation curves of disk galaxies, without need for dark matter. MOND also explains the Tully-Fisher relation, the observed correlation between the infrared luminosity of a disk galaxy and the rotation velocity in the flat region of the rotation curve. See Refs. [7] for other tests and predictions of MOND.

In spite of its success, MOND is not a consistent theory and has some apparent problems. MOND requires dark matter in galaxy clusters, [8], and if this is massive neutrinos the needed neutrino mass is on the edge of being ruled out [9]. There are indications that observations require that the value of a_0 depend on the object being studied [10]. Other possible problems are considered in Refs. [11]. A major drawback of MOND was the lack of a relativistic generalization needed for a consistent MOND cosmological model.

Bekenstein [12] has recently proposed a new covariant field theory which has MOND characteristics in the weak acceleration limit and provides a setting for constructing consistent cosmological models. This model uses three dynamical gravitational fields — a tensor, a vector, and a scalar field — leading to the acronym TeVeS. Several authors have studied different aspects of the TeVeS model, including large-scale structure formation [13], gravitational lensing [14], and the strong gravity regime of the model [15]. Hao and Akhoury [16] have studied a cosmological model and its dynamics in the context of TeVeS , but no exact solution has been found so far.

Whether TeVeS or some generalization is able to provide an accurate model of cosmology is an issue of some current interest. The standard dark energy and dark matter dominated general relativistic cosmological model does a remarkable job on large scales, however, it appears to have trouble fitting some smaller scale observations [5]. TeVeS cosmology on the other hand is only now attracting preliminary attention. In the standard cosmology the Universe goes through two periods of accelerated expansion, at early times during inflation and during the current epoch. Inflation is used to make the Universe homogeneous and to generate the quantum-mechanical fluctuations responsible for observed large-scale structures [17] while supernova and other data indicate that the expansion is speeding up now [5]. It is of interest to understand whether TeVeS cosmology has an accelerated solution that might play a similar role to that in the standard model. A complete and consistent TeVeS cosmological model would be very useful, at

the very least providing stimulus for the development of precision tests that can be used to distinguish the TeVeS predictions from those of standard cosmology, along the lines of observational tests that were developed to distinguish open inflation from the standard flat model [18].

The paper is organized as follows. In Sec. II we present the general TeVeS equations of motion and in Sec. III we specialize these equations to the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker spacetime model. In Sec. IV we present exact solutions of these equations and discuss cosmological implications. We examine the stability of these solutions in Sec. V. Finally, in Sec. VI we summarize our results and present conclusions.

II. TEVES EQUATIONS OF MOTION

TeVeS assumes three dynamical gravitational fields: a tensor metric $g_{\mu\nu}$, a unit norm time-like four-vector U_α, and a scalar ϕ. There is also a nondynamical scalar field σ. The three dynamical fields are connected through the physical metric tensor

$$\tilde{g}_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta} - 2U_\alpha U_\beta \sinh(2\phi) = e^{-2\phi}(g_{\alpha\beta} + U_\alpha U_\beta) - e^{2\phi} U_\alpha U_\beta, \tag{2}$$

where $g_{\alpha\beta}$ is the Einstein metric tensor. Our conventions are those of Bekenstein [12] with metric signature diag$(-1, 1, 1, 1)$.

Following Bekenstein the action of the system is written as

$$S = S_g + S_s + S_v + S_m, \tag{3}$$

where the actions for the tensor field S_g, the two scalar fields S_s, the vector field S_v, and ordinary matter (as opposed to Bekenstein's ϕ, σ, and U_α fields) S_m, are,

$$S_g = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, g^{\alpha\beta} R_{\alpha\beta}, \tag{4}$$

$$S_s = -\frac{1}{2} \int d^4x \sqrt{-g} \left[\sigma^2 h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \frac{G}{2l^2} \sigma^4 F(kG\sigma^2) \right], \tag{5}$$

$$S_v = -\frac{K}{32\pi G} \int d^4x \sqrt{-g} \left[g^{\alpha\beta} g^{\mu\nu} U_{[\alpha,\mu]} U_{[\beta,\nu]} - \frac{2\lambda}{K} \left(g^{\mu\nu} U_\mu U_\nu + 1 \right) \right], \tag{6}$$

$$S_m = \int d^4x \sqrt{-\tilde{g}}\, L(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{|\mu}, \ldots). \tag{7}$$

Here g and $R_{\alpha\beta}$ are the determinant and Ricci tensor constructed from the Einstein metric tensor $g_{\alpha\beta}$, the tensor $h^{\alpha\beta} = g^{\alpha\beta} - U^\alpha U^\beta$ has been used instead of $g^{\alpha\beta}$ in order to avoid acausal dynamical scalar field propagation [19], and f^α represents ordinary matter, including possibly a scalar field or a cosmological constant that drives accelerated expansion or inflation. Also, F is a free dimensionless function, K and k are positive dimensionless coupling constants, l is a constant scale of length, λ is a spacetime dependent Lagrange multiplier, and G is the bare gravitational constant. Square brackets in Eq. (6) denote antisymmetrization, $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$, in what follows round brackets denote symmetrization, $A_{(\mu}B_{\nu)} = A_\mu B_\nu + A_\nu B_\mu$, and the covariant derivative denoted by "$|$" in Eq. (7) is defined in terms of the physical metric tensor.

The equations of motion are obtained by varying the action given in Eq. (3) with respect to U_α, σ, ϕ, $g_{\alpha\beta}$, and λ. These equations are,

$$K(U^{[\alpha;\beta]}{}_{;\beta} + U^\alpha U_\gamma U^{[\gamma;\beta]}{}_{;\beta}) + 8\pi G\sigma^2[U^\beta \phi_{,\beta} g^{\alpha\gamma} \phi_{,\gamma} + U^\alpha (U^\beta \phi_{,\beta})^2] = 8\pi G(1 - e^{-4\phi})[g^{\alpha\mu} U^\beta \tilde{T}_{\mu\beta} + U^\alpha U^\beta U^\gamma \tilde{T}_{\gamma\beta}], \tag{8}$$

$$-\mu F(\mu) - \frac{1}{2}\mu^2 F'(\mu) = kl^2 h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}, \tag{9}$$

$$[\mu(kl^2 h^{\mu\nu} \phi_{,\mu} \phi_{,\nu}) h^{\alpha\beta} \phi_{,\alpha}]_{;\beta} = kG[g^{\alpha\beta} + (1 + e^{-4\phi}) U^\alpha U^\beta] \tilde{T}_{\alpha\beta}, \tag{10}$$

$$G_{\alpha\beta} = 8\pi G[\tilde{T}_{\alpha\beta} + (1 - e^{-4\phi})U^{\mu}\tilde{T}_{\mu(\alpha}U_{\beta)} + \tau_{\alpha\beta}] + \Theta_{\alpha\beta}, \tag{11}$$

$$g^{\mu\nu}U_{\mu}U_{\nu} = -1. \tag{12}$$

Here $\tilde{T}_{\mu\nu}$ is the stress-energy tensor of ordinary matter. Equation (8) is the vector field equation of motion. The prime in Eq. (9) denotes a derivative with respect to $\mu = kG\sigma^2$. Equation (9) determines μ and the nondynamical scalar field σ in terms of $g_{\alpha\beta}$, U_{α}, and ϕ. Equations (10) and (11) are the equations of motion for the dynamical scalar field ϕ and the metric tensor respectively. The Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - Rg_{\alpha\beta}/2$, where R is the Ricci scalar constructed from the metric tensor $g_{\alpha\beta}$. We call the right hand side of Eq. (11) $8\pi G\tilde{T}^{\rm eff}_{\alpha\beta}$.

The tensors $\tau_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ in Eq. (11) are defined as

$$\tau_{\alpha\beta} = \sigma^2\left[\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - U^{\mu}\phi_{,\mu}\left(U_{(\alpha}\phi_{,\beta)} - \frac{1}{2}U^{\nu}\phi_{,\nu}g_{\alpha\beta}\right)\right] - \frac{G}{4l^2}\sigma^4F(\mu)g_{\alpha\beta}, \tag{13}$$

$$\Theta_{\alpha\beta} = K\left(g^{\mu\nu}U_{[\mu,\alpha]}U_{[\nu,\beta]} - \frac{1}{4}g^{\sigma\tau}g^{\mu\nu}U_{[\sigma,\mu]}U_{[\tau,\nu]}g_{\alpha\beta}\right). \tag{14}$$

Assuming that ordinary matter is an ideal fluid, the matter stress-energy tensor in the physical coordinate system is $\tilde{T}_{\alpha\beta} = \tilde{\rho}\tilde{u}_{\alpha}\tilde{u}_{\beta} + \tilde{p}(\tilde{g}_{\alpha\beta} + \tilde{u}_{\alpha}\tilde{u}_{\beta})$, with $\tilde{\rho}$ the energy density, $\tilde{p}$ the pressure, and $\tilde{u}_{\alpha}$ the fluid four-velocity. An equation of state $\tilde{p}(\tilde{\rho})$ must be specified to fully determine this stress-energy tensor. The Bianchi identity $\tilde{T}^{\rm eff}_{\alpha\beta}{}^{;\beta} = 0$, together with Eqs. (8)–(10) and (12), implies stress-energy conservation for ordinary matter, which may be used as an equation of motion for ordinary matter, instead of using part of the tensor equation of motion, Eq. (11). Equation (12), which is a constraint equation derived by varying the action with respect to the Lagrange multiplier λ, forces U_{α} to be time-like with unit norm.

III. SPATIALLY HOMOGENEOUS AND ISOTROPIC EQUATIONS OF MOTION

Observations indicate that on large scales the spatial distribution of matter and radiation are close to isotropic in the mean [3]. This fact motivates consideration of the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker spacetime model. In general relativistic cosmology observations also show that space is close to flat (see, *e.g.*, [5]). Although it is not yet clear if observations also favor vanishing spatial curvature in Bekenstein's model, for simplicity we will consider the spatially-flat Friedmann-Lemaître-Robertson-Walker metric,

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)], \tag{15}$$

where $\tilde{dt} = e^{\phi}dt$ and $\tilde{a} = e^{-\phi}a$ relate the time and scale factor in the physical and Einstein coordinates. The scalar field ϕ depends on time and since there cannot be a preferred spatial direction the vector field U^{μ} must point in the time direction, $U^{\mu} = \delta^{\mu}_{t}$ [12].

In the spatially homogeneous and isotropic model, the tensor field equations of motion obtained from Eq. (11) are

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\tilde{\rho}e^{-2\phi} + \frac{8\pi}{3k}\mu\dot{\phi}^2 + \frac{2\pi}{3k^2l^2}\mu^2F(\mu), \tag{16}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G\tilde{p}e^{-2\phi} - \frac{8\pi\mu}{k}\dot{\phi}^2 + \frac{2\pi\mu^2}{k^2l^2}F(\mu), \tag{17}$$

where an overdot denotes a time derivative. These equations are the analog of the Friedmann equations in general relativistic cosmology. The dynamical scalar field equation of motion obtained from Eq. (10) is

$$\mu\ddot{\phi} + \left(3\mu\frac{\dot{a}}{a} + \dot{\mu}\right)\dot{\phi} + \frac{kG}{2}e^{-2\phi}(\tilde{\rho} + 3\tilde{p}) = 0. \tag{18}$$

Additionally, stress-energy conservation obtained from the Bianchi identity, $\tilde{T}^{\text{eff}\ ;\beta}_{\alpha\beta} = 0$, and Eqs. (8)–(10) and (12) leads to the equation of conservation of stress-energy of ordinary matter,

$$\dot{\tilde{\rho}} = 3\Big(\dot{\phi} - \frac{\dot{a}}{a}\Big)(\tilde{\rho} + \tilde{p}). \tag{19}$$

Consistent with the case of general relativistic cosmology, it may be shown that only three of the four equations (16)–(19) are independent. Note that $U^\mu = \delta^\mu_t$ causes $U^{[\mu;\nu]}$ to vanish, so Eqs. (8) and (12) are identically satisfied. On the other hand, given an expression for the free function $F(\mu)$ (which must be externally specified since there is no theory for its determination), Eq. (9) determines μ and σ in terms of ϕ, U_α, and $g_{\alpha\beta}$. Equations (8), (9), and (12) may be ignored in the following analysis of the spatially homogeneous and isotropic model.

In the following section we look for analytical solutions of the set of field equations (16) and (18) along with (19).

IV. SOLUTIONS

A. Vacuum case

The vacuum case with $\tilde{\rho} = 0 = \tilde{p}$ is a simple case to solve analytically. The Friedmann and scalar field equations become

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3k}\mu\dot{\phi}^2 + \frac{2\pi}{3k^2l^2}\mu^2 F(\mu), \tag{20}$$

$$\mu\ddot{\phi} + \left(3\mu\frac{\dot{a}}{a} + \dot{\mu}\right)\dot{\phi} = 0. \tag{21}$$

Assuming a time independent μ, we find two exact solutions.

Our first solution corresponds to $a = a_0$ and $\phi(t) = \alpha t + \phi_0$, where $\alpha = \left[-\mu F/(4kl^2)\right]^{1/2}$. Since the scale factor is constant, this solution is not very interesting.

Our second solution is the analog of the general relativistic de Sitter solution

$$a(t) = a_0 e^{\pm H_0\, t}, \tag{22}$$

$$\phi(t) = \phi_0, \tag{23}$$

where a_0 and ϕ_0 are constants and $H_0 = \left[2\pi\mu^2 F/(3k^2l^2)\right]^{1/2}$. This solution also depends on the form of the free function $F(\mu)$, which is required to be positive in this case. This solution corresponds to the "slow roll" approximation solution discussed in Ref. [16]. It is interesting that the TeVeS model has an accelerated expansion solution even in the absence of a cosmological constant or a dark energy scalar field.

B. Barotropic fluid case

We assume that the ordinary matter fluid has a barotropic equation of state $\tilde{p} = \omega\tilde{\rho}$, so the stress-energy conservation equation (19) has the solution

$$\tilde{\rho} = \frac{e^{3(1+\omega)\phi}}{a^{3(1+\omega)}}. \tag{24}$$

In order to find an analytical solution of the field equations, we make the ansatz

$$\left[a(t)\right]^{3(1+\omega)} = e^{(1+3\omega)\phi(t)}, \tag{25}$$

so $\tilde{\rho}(t) = \exp(2\phi)$. The Friedmann and scalar field equations, (16) and (18), then become

$$\dot{\phi}^2 \left[\frac{1}{3} \frac{(1+3\omega)^2}{(1+\omega)^2} - \frac{8\pi}{k}\mu \right] = 8\pi G + \frac{2\pi}{k^2 l^2}\mu^2 F(\mu), \tag{26}$$

$$\mu\ddot{\phi} + \frac{1+3\omega}{1+\omega}\mu\dot{\phi}^2 + \dot{\mu}\dot{\phi} + \frac{kG}{2}(1+3\omega) = 0. \tag{27}$$

From Eq. (26) we note that if μ is time independent, $\dot{\phi}$ is time independent and so $\ddot{\phi} = 0$. Taking $\ddot{\phi} = 0$ considerably simplifies Eq. (27), resulting in an analytical expression for the scalar field,

$$\phi(t) = \pm\, \alpha\, t \,+\, \phi_0, \tag{28}$$

where ϕ_0 is a constant of integration and now $\alpha = \left[-kG(1+\omega)/(2\mu)\right]^{1/2}$. Using this solution for $\phi(t)$ in the Friedmann equation (26) we get a relation between the constants ω and μ and the free function $F(\mu)$,

$$\frac{2\pi}{k^2 l^2}\mu^2 F(\mu) = -\frac{kG(1+3\omega)^2}{6\mu(1+\omega)} - 4\pi G(1-\omega). \tag{29}$$

It is important to note that this equation does not give us a function $F(\mu)$, but instead given an $F(\mu)$ it relates μ and ω, or given an $F(\mu)$ and an ω it fixes the value of μ.

From Eqs. (25) and (28) we see that our solution is a de Sitter solution with scale factor

$$a(t) = a_0 e^{\pm H_0 t}, \tag{30}$$

where

$$H_0 = \sqrt{-\frac{kG(1+3\omega)^2}{18\mu(1+\omega)}}, \tag{31}$$

$$a_0 = e^{\frac{1+3\omega}{3(1+\omega)}\phi_0}. \tag{32}$$

Recalling the definition $\mu = kG\sigma^2$, we get $H_0 = (1+3\omega)/\left[-18\sigma^2(1+\omega)\right]^{1/2}$. Since H_0 must be real, our solution must have $\omega < -1$. This is not necessarily a problem, since we are interested in a solution that may be applicable during inflation or the current accelerated expansion epoch. It is interesting that Eq.(30) is a de Sitter solution, independent of the value of ω, provided $\omega < -1$.

Using this solution in the field equations, we find the following useful identities,

$$H_0^2 = \frac{8\pi G}{3}\left[1 + \frac{\mu\alpha^2}{kG} + \frac{\mu^2}{4l^2k^2G}F(\mu)\right], \tag{33}$$

$$3H_0\alpha + \frac{kG}{2\mu}(1+3\omega) = 0. \tag{34}$$

The explicit contribution of the scalar field to the energy density in this model is not easily seen because ordinary matter and the scalar field are entangled, not only in the metric, Eq. (2), but also in the right hand side of Eq. (11). $\tilde{T}_{00} = \tilde{\rho}\, e^{2\phi}$ is the energy density of ordinary matter so $\tilde{T}_{00}^{\rm eff} - \tilde{T}_{00} = -\tilde{\rho}e^{2\phi} + \tilde{\rho}e^{-2\phi} + \mu\dot{\phi}^2/(kG) + \mu^2 F/(4k^2l^2G)$ could be associated with the contribution of the scalar field to the energy density, as in Brans-Dicke-like models [20], but note that in $\tilde{T}_{00}^{\rm eff} - \tilde{T}_{00}$ the stress-energy tensor of ordinary matter also contributes. Of course, $\tilde{T}_{00}^{\rm eff}$ is the right hand side of the Friedmann equation (16), which we rewrite as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{\rm eff}, \tag{35}$$

and similarly the spatial components $\tilde{T}^{\rm eff}_{ij}$ on the right hand side of Eq. (17) may be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho_{\rm eff} + 3p_{\rm eff}\right], \tag{36}$$

where

$$\rho_{\rm eff} = \tilde{\rho} e^{-2\phi} + \frac{\mu}{kG}\dot{\phi}^2 + \frac{\mu^2}{4k^2 l^2 G}F(\mu), \tag{37}$$

$$p_{\rm eff} = \tilde{p} e^{-2\phi} + \frac{\mu}{kG}\dot{\phi}^2 - \frac{\mu^2}{4k^2 l^2 G}F(\mu). \tag{38}$$

$\rho_{\rm eff}$ and $p_{\rm eff}$ obey the energy conservation equation $\dot{\rho}_{\rm eff} = -3(\rho_{\rm eff} + p_{\rm eff})\dot{a}/a$. It may be shown that when $\rho_{\rm eff}$ and $p_{\rm eff}$, Eqs. (37) and (38), are used in this spatially homogeneous conservation equation, we recover the spatially homogeneous part of the scalar field equation of motion, Eq. (18). See, *e.g.*, Ref. [21] for the general relativistic cosmology analog of this result.

Note that in Ref. [16] $\rho_m = \tilde{\rho} e^{-2\phi}$ is defined to be the energy density of ordinary matter, and $\rho_\phi = \mu\dot{\phi}^2/(kG) + \mu^2 F/(4k^2 l^2 G)$ the effective energy density of the scalar field ϕ. However, we have shown here that the energy density of ordinary matter is given by the time-time component of the stress-energy tensor of ordinary matter $\rho_m = \tilde{T}_{00} = \tilde{\rho}\, e^{2\phi}$. The definition of ρ_ϕ in Ref. [16] corresponds to a piece of $\tilde{T}^{\rm eff}_{00} - \tilde{T}_{00}$, that may be associated with the energy density of the scalar field. Thus, the energy densities ρ_m and ρ_ϕ defined in [16] do not correspond to the energy densities of a fluid in the sense that they do not obey an energy conservation equation obtained from the Bianchi identity, which reduces to an equation of motion for the scalar field. These definitions are however, a piece of the effective energy density defined above.

Our solution, Eqs. (28)–(30), implies that $\rho_{\rm eff} = -k(1+3\omega)^2/[48\pi\mu(1+\omega)]$ is positive for $\omega < -1$.

We will examine the stability of our solution in more detail in the following section.

C. Bekenstein's $F(\mu)$ function

The exact solutions that we have obtained are valid for any $F(\mu)$ function, as long as the function satisfies the relation given in Eq. (29) for the solution corresponding to a barotropic equation of state, Sec. IV B, and $F(\mu) > 0$ for the solution corresponding to the vacuum case, Sec. IV A. As pointed out by Bekenstein, [12], there is no theory for the function $F(\mu)$, and based on this freedom and other considerations he picks as an example,

$$F(\mu) = \frac{3}{8\mu}(4 + 2\mu - 4\mu^2 + \mu^3) + \frac{3}{4\mu^2}\ln[(1-\mu)^2]. \tag{39}$$

The cosmologically relevant part of this function lies in the interval $2 < \mu < \infty$. Using this expression, from Eq. (29) we confirm that our solution corresponding to a barotropic perfect fluid is limited to $\omega \le -1$. We also find that $F(\mu) > 0$ for any value of μ in the interval relevant to cosmology, so this form of $F(\mu)$ can also be used in the solution for the vacuum case.

D. Density and deceleration parameters

We rewrite the Friedmann equation, Eq. (16), as

$$1 = \Omega_m + \Omega_\phi, \tag{40}$$

where the density parameters of ordinary matter and the scalar field are,

$$\Omega_m = \frac{8\pi G}{3H^2}\tilde{\rho}\, e^{2\phi}, \tag{41}$$

$$\Omega_\phi = \frac{8\pi G}{3H^2}\Big[-\tilde{\rho}\, e^{2\phi} + \tilde{\rho}\, e^{-2\phi} + \frac{\mu}{kG}\dot{\phi}^2 + \frac{\mu^2}{4k^2 l^2 G}F(\mu)\Big], \tag{42}$$

where $H = \dot{a}/a = \tilde{H} + \dot{\phi}\, e^{-\phi}$, $\tilde{H} = (d\tilde{a}/d\tilde{t})/\tilde{a}$. As in general relativistic cosmology, we use Eqs. (16) and (17) to determine the deceleration parameter, $q = -\ddot{a}/aH^2$, finding

$$q = \frac{4\pi G}{3H^2}\tilde{\rho}\, e^{-2\phi}(1+3\omega) + \frac{8\pi G}{3H^2}\left[\frac{2\mu}{kG}\dot{\phi}^2 - \frac{\mu^2}{4k^2 l^2 G}F(\mu)\right]. \tag{43}$$

For our barotropic perfect fluid solution, Eqs. (28)–(32), Eq. (43) gives $q = -1$ independent of the form of the $F(\mu)$ function and the value of ω. The same result also holds for our solution in the vacuum case (Sec. IV A). Of course, both results follow directly from Eqs. (22) and (30) and the definition of q.

E. Physical variables

Until this point we have been working in the Einstein metric, *i.e.*, in the metric $g_{\mu\nu}$ whose dynamics is governed by the Einstein-Hilbert action. We will call this frame the "Einstein frame". It is important to write the solutions found above in terms of the physical variables, *i.e.*, in the frame of the metric $\tilde{g}_{\mu\nu}$ known as "physical frame" or what Skordis *et al.* [13] called the "matter frame". These metrics are related by Eq. (2), so for the spatially-flat Friedmann-Lemaître-Robertson-Walker metric of Eq. (15) we have $\tilde{a} = e^{-\phi}a$ and $\tilde{dt} = e^{\phi}dt$. In the "physical frame" we define the deceleration parameter in terms of physical variables as

$$q = -\frac{\tilde{a}(\tilde{t})\ddot{\tilde{a}}(\tilde{t})}{\left(\dot{\tilde{a}}(\tilde{t})\right)^2} \tag{44}$$

where now an overdot denotes differentiation with respect to physical time $\tilde{t}$.

- Vacuum case.

  The solution of the vacuum case in the Einstein frame, Eqs. (22) and (23), corresponds to the physical frame solution

$$\tilde{a}(\tilde{t}) = \tilde{a}_0 e^{\pm\tilde{H}_0\tilde{t}}, \tag{45}$$

  where $\tilde{a}_0 = a_0 e^{-\phi_0}$ and $\tilde{H}_0 = H_0 e^{-\phi_0}$. The vacuum solution in the physical variables is also a de Sitter solution with $q = -1$, as can be seen from Eq. (44).

- Barotropic ideal fluid case.

  We have found that the solution for a barotropic perfect fluid in the Einstein frame is $a(t) = a_0 e^{\pm H_0 t}$ and $\phi(t) = \pm\alpha t + \phi_0$, where H_0 and α are given by Eqs. (31) and (28). Rewriting this solution in terms of the physical variables we get

$$\tilde{a}(\tilde{t}) = \tilde{a}_0\, \tilde{t}^{H_0/\alpha - 1} = \tilde{a}_0\, \tilde{t}^{-2/3(1+\omega)}, \tag{46}$$

$$\phi(\tilde{t}) = \ln(\alpha\tilde{t}) \tag{47}$$

  where $\tilde{a}_0 = a_0\, e^{-\phi_0 H_0/\alpha}(\alpha)^{(H_0/\alpha - 1)} = [-kG(1+\omega)/(2\mu)]^{-1/3(1+\omega)}$. Note that the contracting solution in the Einstein frame does not have physical meaning in the physical variables. As we can see, the de Sitter solution in the Einstein frame for a barotropic ideal fluid is transformed to a power law solution in the physical frame. Since $\omega < -1$ the solution is expanding.

  Using Eq. (44) it is straightforward to check that the deceleration parameter in the physical frame for the solution (46) is $q = -(5+3\omega)/2$, then $q < 0$ implies $\omega > -5/3$. Since $\omega < -1$ this solution expands in an accelerated way for ω in the interval of values $-5/3 < \omega < -1$. Such solutions have been discussed in the context of power law inflation [24].

V. STABILITY OF THE SOLUTIONS

In Sec. IV we have found exact solutions for the vacuum case, Eqs. (22) and (23), and for the case with a barotropic ideal fluid, Eqs. (28) and (30). The purpose of this section is to see if these solutions are stable with respect to small perturbations.

A complete analysis will require solving the TeVeS cosmological linear perturbation equations (see Refs. [13, 22]) for small departures from the spatially homogeneous and isotropic background solutions that we have derived in the previous section. In our preliminary exploration here we will focus only on very large scale perturbations, using instead the method developed in Ref. [23].

We first consider the barotropic ideal fluid case. Following the method of Ref. [23] we make the change of variables,

$$\begin{aligned} a(t) &= a_e(t)v(t), \\ \phi(t) &= \phi_e(t) + \psi(t), \\ \tilde{\rho}(t) &= \tilde{\rho}_e(t) + \eta(t). \end{aligned} \tag{48}$$

Here $v(t)-1$, $\psi(t)$, and $\eta(t)$ are small perturbations about the spatially homogeneous solution, denoted here as a_e, ϕ_e, and ρ_e, and given in Eqs. (30), (28), and (24). Defining two "speeds" $s = \dot{\psi}$ and $r = \dot{v}$ (note that $\tilde{\rho}$ obeys a first order differential equation and so we do not need a "speed" for it), the equations of motion become,

$$\begin{aligned} \dot{\psi} &= s, \\ \dot{s} &= -\frac{3\alpha}{v\mu}r - \frac{\dot{\mu}}{\mu}\alpha - \left(3H_0 + 3\frac{r}{v} + \frac{\dot{\mu}}{\mu}\right)s - \frac{kG(1+3\omega)}{2\mu}e^{-2\phi_e}\left(e^{-2\psi}(\tilde{\rho}_e+\eta) - \tilde{\rho}_e\right), \\ \dot{v} &= r, \\ \dot{r} &= -2H_0 r - \frac{4\pi G}{3}(1+3\omega)e^{-2\phi_e}\left(e^{-2\psi}(\tilde{\rho}_e+\eta) - \tilde{\rho}_e\right)v - \frac{4\pi G}{3}v\left(\mathcal{F}(\alpha+s) - \mathcal{F}(\alpha)\right), \\ \dot{\eta} &= -3\left(\frac{r}{v} - s\right)(1+\omega)\tilde{\rho}_e - 3\left(H_0 + \frac{r}{v} - \alpha - s\right)\eta, \end{aligned} \tag{49}$$

where $\mathcal{F} = \mu\dot{\phi}^2/(kG) + \mu^2 F(\mu)/(4l^2k^2G)$.

The vector field U^α contributes to the initial set of equations (8)–(12), but it does not contribute to the spatially homogeneous cosmological equations (16)–(19), because there is no preferred spatial direction on cosmological scales and so $U^\alpha = \delta^\alpha_t$. To simplify the analysis we do not consider a perturbation of U^α here.

$\mu(\dot{\phi})$ is a function of the scalar field's time derivative and is time independent for our background solutions $\mu = \mu(\alpha)$. When we perturb the scalar field, $\mu = \mu(\alpha+s)$ becomes time dependent. Since we are interested in linear perturbation theory, we expand it in a Taylor series with respect to the small parameter s, and we keep only terms up to first order $\mu(\alpha+s) = \mu(\alpha) + s\, d\mu/d\alpha$.

The critical point of Eqs. (49) is $(\psi_0, s_0, v_0, r_0, \eta_0) = (0, 0, \bar{v}, 0, 0)$, where $\bar{v}$ is an arbitrary constant which corresponds to the freedom in rescaling a_0. This point of phase space is where all time derivatives in Eqs. (49) vanish. Perturbing Eqs. (49) around the critical point, $(\psi, s, v, r, \eta) = (\psi_0+\psi_1, s_0+s_1, v_0+v_1, r_0+r_1, \eta_0+\eta_1)$, the linear perturbation equations are,

$$\begin{aligned} \dot{\psi}_1 &= s_1, \\ \dot{s}_1 &= \frac{kG}{\mu_0}(1+3\omega)\psi_1 - 3H_0 s_1 - \frac{3\alpha}{\bar{v}\mu_0}r_1 - \frac{kG}{2\mu_0}(1+3\omega)e^{-2\phi_e}\eta_1, \\ \dot{v}_1 &= r_1, \\ \dot{r}_1 &= \frac{8\pi G}{3}(1+3\omega)\bar{v}\psi_1 - \frac{4\pi G}{3}\frac{d\mathcal{F}}{d\alpha}\bar{v}s_1 - 2H_0 r_1 - \frac{4\pi G}{3}(1+3\omega)e^{-2\phi_e}\bar{v}\eta_1, \\ \dot{\eta}_1 &= 3(1+\omega)\tilde{\rho}_e s_1 - 3\frac{(1+\omega)\tilde{\rho}_e}{\bar{v}}r_1 - 3(H_0-\alpha)(1+\omega)\eta_1, \end{aligned} \tag{50}$$

where $\mu_0 = \mu(\alpha)$.

In order to compute the eigenvalues of the matrix defined by these equations, we need to specify the function $F(\mu)$ and the values of the constants of the TeVeS model. As an example, we consider Bekenstein's [12] $F(\mu)$ function, Eq. (39), for some values of μ in the cosmological regime $2 < \mu < \infty$. We choose two set of arbitrary values for the

parameters of the model, $k = l = G = 1$ and $k = 0.1$, $l = 10$, $G = 1$. Also, we consider three illustrative values for $\omega = -1.02$, -1.05, and -1.08. Numerical computation shows that the expanding solution, $a(t) = a_0 e^{H_0 t}$, is stable for small values of μ ($\mu < 19$ when $\omega = -1.02$, $\mu < 13$ when $\omega = -1.05$, and $\mu < 11$ when $\omega = -1.08$) for $k = l = G = 1$. For the second set of parameters $k = 0.1$, $l = 10$, $G = 1$ we find that the expanding solution is stable for small values of μ ($\mu < 11$) for all three values of ω that we consider. The contracting solution, $a(t) = a_0 e^{-H_0 t}$, on the other hand, is not stable for any μ in the cosmological allowed range, independent of the values of the other model parameters we consider. In the limit of small deviation from $\omega = -1$, analytical computations show that the expanding solution $a(t) = a_0 e^{H_0 t}$ is stable. Clearly, the stability of the solution depends sensitively on the form of $F(\mu)$ and the values of k, l, G, and μ. We note that even though $\omega < -1$, this does not necessarily imply instability.

The expanding solution for the vacuum case, $a(t) = a_0 e^{H_0 t}$ with $\phi(t) = \phi_0$, is stable with linear perturbation eigenvalues $\lambda_1 = -2H_0$, $\lambda_2 = -3H_0$. The contracting solution, $a(t) = a_0 e^{-H_0 t}$, is unstable with eigenvalues $\lambda_1 = 2H_0$, $\lambda_2 = 3H_0$. These results hold for any $F(\mu) > 0$ function.

We have calculated analytically the eigenvalues of small perturbations in the limit of general relativity. We find that for both the vacuum and barotropic equation of state cases, the expanding solutions $a(t) = a_0 e^{H_0 t}$ are stable, as well known.

VI. CONCLUSIONS

We have found two homogeneous and isotropic cosmological solutions of the TeVeS model, a relativistic generalization of MOND. In the vacuum case, the analytical solutions in both, Einstein and physical frames, are de Sitter solutions with $q = -1$, and the scalar field is a constant $\phi(t) = \phi_0$. For a barotropic ideal fluid we find a de Sitter solution in the Einstein frame, which is transformed to a power law expanding solution in the physical frame. Accelerated expansion for this solution occurs for $-5/3 < \omega < -1$. The scalar field in this case evolves logarithmically with the physical time, $\phi(\tilde{t}) = \ln(\alpha\tilde{t})$.

The free function $F(\mu)$ plays an important role in the TeVeS model. However, we have found that for vacuum case the only restriction is $F(\mu) > 0$, while for the barotropic perfect fluid case our solutions are valid for a family of $F(\mu)$ functions that obey the ω dependent relationship given in Eq. (29).

In the vacuum case the expanding solution $a(t) = a_0 e^{Ht}$ is stable for any $F(\mu) > 0$ function. In the more general case of a barotropic ideal fluid, we have studied stability for a few values of the parameters and constants of the theory and have assumed Bekenstein's $F(\mu)$ function. Typically, the expanding solution, $a = a_0 e^{Ht}$ in Einstein frame, is stable for small values of μ, and becomes unstable for large values of μ. Since the solutions in Einstein frame and physical frame are related through transformation of variables, the corresponding solution in the physical frame $\tilde{a}(\tilde{t}) = \tilde{a}_0\, \tilde{t}^{H_0/\alpha - 1}$ has the same stability properties.

The accelerated expansion TeVeS cosmological solutions we have found might prove helpful when trying to incorporate inflation or the present epoch of accelerated expansion in the TeVeS scenario.

Acknowledgments

We are indebted to the referee for very useful comments which helped us to clarify and improve our results. We acknowledge support from DOE EPSCoR grant DE-FG02-00ER45824 and DOE grant DE-FG03-99ER41093.

[1] F. Zwicky, Helv. Phys. Acta **6**, 110 (1933), Astrophys. J. **86**, 217 (1937); S. Smith, Astrophys. J. **83**, 23 (1936); H. W. Babcock, Lick Obs. Bull. **19**, 41 (1939); J. H. Oort, Astrophys. J. **91**, 273 (1940); H. C. van de Hulst, E. Raimond, and H. van Woerden, Bull. Astron. Inst. Netherlands **14**, 1 (1957); V. C. Rubin and W. K. Ford, Jr., Astrophys. J. **159**, 379 (1970); J. Einasto, A. Kaasik, and E. Saar, Nature **250**, 309 (1974); J. P. Ostriker, P. J. E. Peebles, and A. Yahil, Astrophys. J. Lett. **193**, L1 (1974).

[2] V. Trimble, Annu. Rev. Astron. Astrophys. **25**, 425 (1987); K. M. Ashman, Publ. Astron. Soc. Pac. **104**, 1109 (1992).

[3] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993).

[4] G. Bertone, D. Hooper, and J. Silk, Phys. Rept. **405**, 279 (2005).

[5] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).

[6] M. Milgrom, Astrophys. J. **270**, 365 (1983).

[7] J. Bekenstein and M. Milgrom, Astrophys. J. **286**, 7 (1984); R. Sanders and S. McGaugh, Annu. Rev. Astron. Astrophys. **40**, 263 (2002); M. Milgrom and R. Sanders, Astrophys. J. Lett. **599**, L25 (2003); H. Zhao and L. Tian, astro-ph/0511754; C. W. Stubbs and A. Garg, astro-ph/0512067.

[8] R. Sanders, Mon. Not. R. Astron. Soc. **342**, 901 (2003); E. Pointecouteau and J. Silk, astro-ph/0505017.
[9] Ø. Elgarø y and O. Lahav, New J. Phys. **7**, 61 (2005); T. Kahniashvili, E. von Toerne, N. Arhipova, and B. Ratra, Phys. Rev. D **71**, 125009 (2005); S. Hannestad, astro-ph/0511595.
[10] B. Famaey and J. Binney, Mon. Not. R. Astron. Soc. **363**, 603 (2005).
[11] H. Zhao, astro-ph/0508635; E. Łokas, G. Mamon, and F. Prada, astro-ph/0508668.
[12] J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004), Erratum: Phys. Rev. D **71**, 069901 (2005).
[13] C. Skordis, D. Mota, P. Ferreira, and C. Boehm, Phys. Rev. Lett. **96**, 011301 (2006).
[14] M. Chiu, C. Ko, and Y. Tian, astro-ph/0507332; H. Zhao, D. Bacon, A. Taylor, and K. Horne, astro-ph/0509590.
[15] D. Giannios, Phys. Rev. D **71**, 103511 (2005).
[16] J. G. Hao and R. Akhoury, astro-ph/0504130.
[17] S. W. Hawking, Phys. Lett. B **115**, 295 (1982); A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982); A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner Phys. Rev. D **28**, 679 (1983); W. Fischler, B. Ratra, and L. Susskind, Nucl. Phys. B **259**, 730 (1985).
[18] J.R. Gott, III, Nature **295**, 304 (1982); B. Ratra and P. J. E. Peebles, Phys. Rev. D **52**, 1837 (1995); P. Mukherjee, et al., Int. J. Mod. Phys. A **18**, 4933 (2003).
[19] J. D. Bekenstein, HEP PoS (**jhw2004**), 012 (2004).
[20] R. V. Wagoner, Phys. Rev. D **1**, 3209 (1970).
[21] B. Ratra, Phys. Rev. D **43**, 3802 (1991).
[22] C. Skordis, astro-ph/0511591.
[23] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[24] L. F. Abbott and M. B. Wise, Nucl. Phys. B **244**, 541 (1984); F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985); B. Ratra, Phys. Rev. D **40**, 3939 (1989).